

HOMOLOGY OF ARTINIAN AND MATLIS REFLEXIVE MODULES, I

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ABSTRACT. Let R be a commutative local noetherian ring, and let L and L' be R -modules. We investigate the properties of the functors $\mathrm{Tor}_i^R(L, -)$ and $\mathrm{Ext}_R^i(L, -)$. For instance, we show the following:

- (a) if L is artinian and L' is noetherian, then $\mathrm{Hom}_R(L, L')$ has finite length;
- (b) if L and L' are artinian, then $L \otimes_R L'$ has finite length;
- (c) if L and L' are artinian, then $\mathrm{Tor}_i^R(L, L')$ is artinian, and $\mathrm{Ext}_R^i(L, L')$ is noetherian over the completion \widehat{R} ; and
- (d) if L is artinian and L' is Matlis reflexive, then $\mathrm{Ext}_R^i(L, L')$, $\mathrm{Ext}_R^i(L', L)$, and $\mathrm{Tor}_i^R(L, L')$ are Matlis reflexive.

Also, we study the vanishing behavior of these functors, and we include computations demonstrating the sharpness of our results.

INTRODUCTION

Throughout this paper, let R be a commutative noetherian local ring with maximal ideal \mathfrak{m} and residue field $k = R/\mathfrak{m}$. The \mathfrak{m} -adic completion of R is denoted \widehat{R} , the injective hull of k is $E = E_R(k)$, and the Matlis duality functor is $(-)^{\vee} = \mathrm{Hom}_R(-, E)$.

This paper is concerned, in part, with properties of the functors $\mathrm{Hom}_R(A, -)$ and $A \otimes_R -$, where A is an artinian R -module. For instance, the following result is contained in Corollaries 2.12 and 3.9.

Theorem 1. *Let A , A' and N be R -modules such that A and A' are artinian and N is noetherian. Then the modules $\mathrm{Hom}_R(A, N)$ and $A \otimes_R A'$ have finite length.*

This follows from the fact that $A \otimes_R A'$ can be described as the tensor product of two finite length modules, and an analogous description holds for $\mathrm{Hom}_R(A, N)$.

In light of Theorem 1, it is natural to investigate the properties of $\mathrm{Ext}_R^i(A, -)$ and $\mathrm{Tor}_i^R(A, -)$. In general, the modules $\mathrm{Ext}_R^i(A, N)$ and $\mathrm{Tor}_i^R(A, A')$ will not have finite length. However, we have the following; see Theorems 2.2 and 3.1.

Theorem 2. *Let A be an artinian R -module, and let $i \geq 0$. Let L and L' be R -modules such that $\mu_R^i(L)$ and $\beta_i^R(L')$ are finite. Then $\mathrm{Ext}_R^i(A, L)$ is a noetherian \widehat{R} -module, and $\mathrm{Tor}_i^R(A, L')$ is artinian.*

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In this result, we are using the i th Bass number $\mu_R^i(L) := \text{len}_R(\text{Ext}_R^i(k, L))$ and the i th Betti number $\beta_i^R(L') := \text{len}_R(\text{Tor}_i^R(k, L'))$. For instance, these are both finite for all i when L and L' are either artinian or noetherian. In particular, when A and A' are artinian, Theorem 2 implies that $\text{Ext}_R^i(A, A')$ is a noetherian \widehat{R} -module. The next result, contained in Theorem 4.3, gives another explanation for this fact.

Theorem 3. *Let A and A' be artinian R -modules, and let $i \geq 0$. Then there is an isomorphism $\text{Ext}_R^i(A, A') \cong \text{Ext}_{\widehat{R}}^i(A^\vee, A^\vee)$. Hence, there are noetherian \widehat{R} -modules N and N' such that $\text{Ext}_R^i(A, A') \cong \text{Ext}_{\widehat{R}}^i(N, N')$.*

This result proves useful for studying the vanishing of $\text{Ext}_R^i(A, A')$, since the vanishing of $\text{Ext}_{\widehat{R}}^i(N, N')$ is somewhat well understood.

Our next result shows how extra conditions on the modules in Theorem 2 imply that $\text{Ext}_R^i(A, L)$ and $\text{Tor}_i^R(A, L')$ are Matlis reflexive; see Corollaries 2.4 and 3.3.

Theorem 4. *Let A , L , and L' be R -modules such that A is artinian. Assume that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ and $R/(\text{Ann}_R(A) + \text{Ann}_R(L'))$ are complete. Given an index $i \geq 0$ such that $\mu_R^i(L)$ and $\beta_i^R(L')$ are finite, the modules $\text{Ext}_R^i(A, L)$ and $\text{Tor}_i^R(A, L')$ are Matlis reflexive.*

A key point in the proof of this theorem is a result of Belshoff, Enochs, and García Rozas [3]: An R -module M is Matlis reflexive if and only if it is mini-max and $R/\text{Ann}_R(M)$ is complete. Here M is *mini-max* when M has a noetherian submodule N such that M/N is artinian. In particular, noetherian modules are mini-max, as are artinian modules.

The last result singled out for this introduction describes the Matlis dual of $\text{Ext}_R^i(M, M')$ in some special cases. It is contained in Corollary 4.11.

Theorem 5. *Let M and M' be mini-max R -modules, and fix an index $i \geq 0$. If either M or M' is Matlis reflexive, then $\text{Ext}_R^i(M, M')^\vee \cong \text{Tor}_i^R(M, M'^\vee)$.*

We do not include a description of the Matlis dual of $\text{Tor}_i^R(M, M')$, as a standard application of Hom-tensor adjointness shows that $\text{Tor}_i^R(M, M')^\vee \cong \text{Ext}_R^i(M, M'^\vee)$.

Many of our results generalize to the non-local setting. As this generalization requires additional tools, we treat it separately in [9].

1. BACKGROUND MATERIAL AND PRELIMINARY RESULTS

Torsion Modules.

Definition 1.1. Let \mathfrak{a} be a proper ideal of R . We denote the \mathfrak{a} -adic completion of R by $\widehat{R}^\mathfrak{a}$. Given an R -module L , set $\Gamma_\mathfrak{a}(L) = \{x \in L \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}$. We say that L is \mathfrak{a} -torsion if $L = \Gamma_\mathfrak{a}(L)$. We set $\text{Supp}_R(L) = \{\mathfrak{p} \in \text{Spec}(R) \mid L_\mathfrak{p} \neq 0\}$.

Fact 1.2. Let \mathfrak{a} be a proper ideal of R , and let L be an \mathfrak{a} -torsion R -module.

- (a) Every artinian R -module is \mathfrak{m} -torsion. In particular, the module E is \mathfrak{m} -torsion.
- (b) We have $\text{Supp}_R(L) \subseteq V(\mathfrak{a})$. Hence, if L is \mathfrak{m} -torsion, then $\text{Supp}_R(L) \subseteq \{\mathfrak{m}\}$.
- (c) The module L has an $\widehat{R}^\mathfrak{a}$ -module structure that is compatible with its R -module structure, as follows. For each $x \in L$, fix an exponent n such that $\mathfrak{a}^n x = 0$. For each $r \in \widehat{R}^\mathfrak{a}$, the isomorphism $\widehat{R}^\mathfrak{a}/\mathfrak{a}^n \widehat{R}^\mathfrak{a} \cong R/\mathfrak{a}^n$ provides an element $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n \widehat{R}^\mathfrak{a}$, and we set $rx := r_0 x$.

(d) If R/\mathfrak{a} is complete, then $\widehat{R}^{\mathfrak{a}}$ is naturally isomorphic to \widehat{R} .

Lemma 1.3. *Let \mathfrak{a} be a proper ideal of R , and let L be an \mathfrak{a} -torsion R -module.*

- (a) *A subset $Z \subseteq L$ is an R -submodule if and only if it is an $\widehat{R}^{\mathfrak{a}}$ -submodule.*
- (b) *The module L is noetherian over R if and only if it is noetherian over $\widehat{R}^{\mathfrak{a}}$.*

Proof. (a) Every $\widehat{R}^{\mathfrak{a}}$ -submodule of L is an R -submodule by restriction of scalars. Conversely, fix an R -submodule $Z \subseteq L$. Since L is \mathfrak{a} -torsion, so is Z , and Fact 1.2(c) implies that Z is an $\widehat{R}^{\mathfrak{a}}$ -submodule.

(b) The set of R -submodules of L equals the set of $\widehat{R}^{\mathfrak{a}}$ -submodules of L , so they satisfy the ascending chain condition simultaneously. \square

Lemma 1.4. *Let \mathfrak{a} be a proper ideal of R , and let L be an \mathfrak{a} -torsion R -module.*

- (a) *The natural map $L \rightarrow \widehat{R}^{\mathfrak{a}} \otimes_R L$ is an isomorphism.*
- (b) *The left and right $\widehat{R}^{\mathfrak{a}}$ -module structures on $\widehat{R}^{\mathfrak{a}} \otimes_R L$ are the same.*

Proof. The natural map $L \rightarrow \widehat{R}^{\mathfrak{a}} \otimes_R L$ is injective, as $\widehat{R}^{\mathfrak{a}}$ is faithfully flat over R . To show surjectivity, it suffices to show that each generator $r \otimes x \in \widehat{R}^{\mathfrak{a}} \otimes_R L$ is of the form $1 \otimes x'$ for some $x' \in L$. Let $n \geq 1$ such that $\mathfrak{a}^n x = 0$, and let $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n \widehat{R}^{\mathfrak{a}}$. It follows that $r \otimes x = r_0 \otimes x = 1 \otimes (r_0 x)$, and this yields the conclusion of part (a). This also proves (b) because $1 \otimes (r_0 x) = 1 \otimes (rx)$. \square

Lemma 1.5. *Let \mathfrak{a} be a proper ideal of R , and let L and L' be R -modules such that L is \mathfrak{a} -torsion.*

- (a) *If L' is \mathfrak{a} -torsion, then $\mathrm{Hom}_R(L, L') = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$; thus $L^\vee = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, E)$.*
- (b) *One has $\mathrm{Hom}_R(L, L') \cong \mathrm{Hom}_R(L, \Gamma_{\mathfrak{a}}(L')) = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, \Gamma_{\mathfrak{a}}(L'))$.*

Proof. (a) It suffices to verify the inclusion $\mathrm{Hom}_R(L, L') \subseteq \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$. Let $x \in L$ and $r \in \widehat{R}^{\mathfrak{a}}$, and fix $\psi \in \mathrm{Hom}_R(L, L')$. Let $n \geq 1$ such that $\mathfrak{a}^n x = 0$ and $\mathfrak{a}^n \psi(x) = 0$. Choose an element $r_0 \in R$ such that $r - r_0 \in \mathfrak{a}^n \widehat{R}^{\mathfrak{a}}$. It follows that $\psi(rx) = \psi(r_0 x) = r_0 \psi(x) = r \psi(x)$; hence $\psi \in \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(L, L')$.

(b) For each $f \in \mathrm{Hom}_R(L, L')$, one has $\mathrm{Im}(f) \subseteq \Gamma_{\mathfrak{a}}(L')$. This yields the desired isomorphism, and the equality is from part (a). \square

A Natural Map from $\mathrm{Tor}_i^R(L, L'^\vee)$ to $\mathrm{Ext}_R^i(L, L')^\vee$.

Definition 1.6. Let L be an R -module, and let J be an R -complex. The *Hom-evaluation* morphism

$$\theta_{LJE}: L \otimes_R \mathrm{Hom}_R(J, E) \rightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(L, J), E)$$

is given by $\theta_{LJE}(l \otimes \psi)(\phi) = \psi(\phi(l))$.

Remark 1.7. Let L and L' be R -modules, and let J be an injective resolution of L' . Using the notation $(-)^\vee$, we have $\theta_{LJE}: L \otimes_R J^\vee \rightarrow \mathrm{Hom}_R(L, J)^\vee$. The complex J^\vee is a flat resolution of L'^\vee ; see, e.g., [6, Theorem 3.2.16]. This explains the first isomorphism in the following sequence:

$$\mathrm{Tor}_i^R(L, L'^\vee) \xrightarrow{\cong} \mathrm{H}_i(L \otimes_R J^\vee) \xrightarrow{\mathrm{H}_i(\theta_{LJE})} \mathrm{H}_i(\mathrm{Hom}_R(L, J)^\vee) \xrightarrow{\cong} \mathrm{Ext}_R^i(L, L')^\vee.$$

For the second isomorphism, the exactness of $(-)^\vee$ implies that $\mathrm{H}_i(\mathrm{Hom}_R(L, J)^\vee) \cong \mathrm{H}^i(\mathrm{Hom}_R(L, J))^\vee \cong \mathrm{Ext}_R^i(L, L')^\vee$.

Definition 1.8. Let L and L' be R -modules, and let J be an injective resolution of L' . The R -module homomorphism

$$\Theta_{LL'}^i: \operatorname{Tor}_i^R(L, L'^\vee) \rightarrow \operatorname{Ext}_R^i(L, L')^\vee$$

is defined to be the composition of the the maps displayed in Remark 1.7.

Remark 1.9. Let L , L' , and N be R -modules such that N is noetherian. It is straightforward to show that the map $\Theta_{LL'}^i$ is natural in L and in L' .

The fact that E is injective implies that $\Theta_{NL'}^i$ is an isomorphism; see [15, Lemma 3.60]. This explains the first of the following isomorphisms:

$$\operatorname{Ext}_R^i(N, L')^\vee \cong \operatorname{Tor}_i^R(N, L'^\vee) \quad \operatorname{Tor}_i^R(L, L')^\vee \cong \operatorname{Ext}_R^i(L, L'^\vee).$$

The second isomorphism is a consequence of Hom-tensor adjointness,

Numerical Invariants.

Definition 1.10. Let L be an R -module. For each integer i , the i th *Bass number* of L and the i th *Betti number* of L are respectively

$$\mu_R^i(L) = \operatorname{len}_R(\operatorname{Ext}_R^i(k, L)) \quad \beta_i^R(L) = \operatorname{len}_R(\operatorname{Tor}_i^R(k, L))$$

where $\operatorname{len}_R(L')$ denotes the length of an R -module L' .

Remark 1.11. Let L be an R -module.

- (a) If I is a minimal injective resolution of L , then for each index $i \geq 0$ such that $\mu_R^i(L) < \infty$, we have $I^i \cong E^{\mu_R^i(L)} \oplus J^i$ where J^i does not have E as a summand, that is, $\Gamma_{\mathfrak{m}}(J^i) = 0$; see, e.g., [12, Theorem 18.7]. Similarly, the Betti numbers of a noetherian module are the ranks of the free modules in a minimal free resolution. The situation for Betti numbers of non-noetherian modules is more subtle; see, e.g., Lemma 1.19.
- (b) Then $\mu_R^i(L) < \infty$ for all $i \geq 0$ if and only if $\beta_i^R(L) < \infty$ for all $i \geq 0$; see [10, Proposition 1.1].

When $\mathfrak{a} = \mathfrak{m}$, the next invariants can be interpreted in terms of (non)vanishing Bass and Betti numbers.

Definition 1.12. Let \mathfrak{a} be an ideal of R . For each R -module L , set

$$\begin{aligned} \operatorname{depth}_R(\mathfrak{a}; L) &= \inf\{i \geq 0 \mid \operatorname{Ext}_R^i(R/\mathfrak{a}, L) \neq 0\} \\ \operatorname{width}_R(\mathfrak{a}; L) &= \inf\{i \geq 0 \mid \operatorname{Tor}_i^R(R/\mathfrak{a}, L) \neq 0\}. \end{aligned}$$

We write $\operatorname{depth}_R(L) = \operatorname{depth}_R(\mathfrak{m}; L)$ and $\operatorname{width}_R(L) = \operatorname{width}_R(\mathfrak{m}; L)$.

Part (b) of the next result is known. We include it for ease of reference.

Lemma 1.13. *Let L be an R -module, and let \mathfrak{a} be an ideal of R .*

- (a) *Then $\operatorname{width}_R(\mathfrak{a}; L) = \operatorname{depth}_R(\mathfrak{a}; L^\vee)$ and $\operatorname{width}_R(\mathfrak{a}; L^\vee) = \operatorname{depth}_R(\mathfrak{a}; L)$.*
- (b) *For each index $i \geq 0$ we have $\beta_i^R(L) = \mu_R^i(L^\vee)$ and $\beta_i^R(L^\vee) = \mu_R^i(L)$.*
- (c) *$L = \mathfrak{a}L$ if and only if $\operatorname{depth}_R(\mathfrak{a}; L^\vee) > 0$.*
- (d) *$L^\vee = \mathfrak{a}(L^\vee)$ if and only if $\operatorname{depth}_R(\mathfrak{a}; L) > 0$.*
- (e) *$\operatorname{depth}_R(\mathfrak{a}; L) > 0$ if and only if \mathfrak{a} contains a non-zero-divisor for L .*

Proof. Part (a) is from [7, Proposition 4.4], and part (b) follows directly from this.

(c)–(d) These follow from part (a) since $L = \mathfrak{a}L$ if and only if $\text{width}_R(\mathfrak{a}; L) > 0$.

(e) By definition, we need to show that $\text{Hom}_R(R/\mathfrak{a}, L) = 0$ if and only if \mathfrak{a} contains a non-zero-divisor for L . One implication is explicitly stated in [5, Proposition 1.2.3(a)]. One can prove the converse like [5, Proposition 1.2.3(b)], using the fact that R/\mathfrak{a} is finitely generated. \square

The next result characterizes artinian modules in terms of Bass numbers.

Lemma 1.14. *Let L be an R -module. The following conditions are equivalent:*

- (i) L is an artinian R -module;
- (ii) L is an artinian \widehat{R} -module;
- (iii) $\widehat{R} \otimes_R L$ is an artinian \widehat{R} -module; and
- (iv) L is \mathfrak{m} -torsion and $\mu_R^0(L) < \infty$.

Proof. (i) \iff (iv) If M is artinian over R , then it is \mathfrak{m} -torsion by Fact 1.2(a), and we have $\mu_R^0(L) < \infty$ by [6, Theorem 3.4.3]. For the converse, assume that L is \mathfrak{m} -torsion and $\mu^0 = \mu_R^0(L) < \infty$. Since L is \mathfrak{m} -torsion, so is $E_R(L)$. Thus, we have $E_R(L) \cong E^{\mu^0}$, which is artinian since $\mu^0 < \infty$. Since L is a submodule of the artinian module $E_R(L)$, it is also artinian.

To show the equivalence of the conditions (i)–(iii), first note that each of these conditions implies that L is \mathfrak{m} -torsion. (For condition (iii), use the monomorphism $L \rightarrow \widehat{R} \otimes_R L$.) Thus, for the rest of the proof, we assume that L is \mathfrak{m} -torsion.

Because of the equivalence (i) \iff (iv), it suffices to show that

$$\mu_R^0(L) = \mu_{\widehat{R}}^0(L) = \mu_{\widehat{R}}^0(\widehat{R} \otimes_R L).$$

These equalities follow from the next isomorphisms

$$\text{Hom}_R(k, L) \cong \text{Hom}_{\widehat{R}}(k, L) \cong \text{Hom}_{\widehat{R}}(k, \widehat{R} \otimes_R L)$$

which are from Lemmas 1.5(a) and 1.4, respectively. \square

Lemma 1.15. *Let L be an R -module.*

- (a) *The module L is noetherian over R if and only if L^\vee is artinian over R .*
- (b) *If L^\vee is noetherian over R or over \widehat{R} , then L is artinian over R .*
- (c) *Let \mathfrak{a} be a proper ideal of R such that R/\mathfrak{a} is complete. If L is \mathfrak{a} -torsion, then L is artinian over R if and only if L^\vee is noetherian over R .*

Proof. (a) Assume first that L is noetherian. Then L is a homomorphic image of R^b for some integer $b \geq 0$. It follows that L^\vee is isomorphic to a submodule of the artinian module $(R^b)^\vee \cong E^b$, so L^\vee is artinian.

For the converse, assume that L^\vee is artinian, and fix an ascending chain $L_1 \subseteq L_2 \subseteq \cdots$ of submodules of L . Dualize the surjections $L \twoheadrightarrow L/L_1 \twoheadrightarrow L/L_2 \twoheadrightarrow \cdots$ to obtain a sequence of monomorphisms $\cdots \hookrightarrow (L/L_2)^\vee \hookrightarrow (L/L_1)^\vee \hookrightarrow L^\vee$. The corresponding descending chain of submodules must stabilize since L^\vee is artinian, and it follows that the original chain $L_1 \subseteq L_2 \subseteq \cdots$ of submodules of L also stabilizes. Thus L is noetherian.

(b) Argue as in part (a).

(c) Assume that L is \mathfrak{a} -torsion. One implication is from part (b). For the converse, assume that L is artinian over R . Lemma 1.14 shows that $L^\vee = \text{Hom}_{\widehat{R}}(L, E)$ is artinian over \widehat{R} ; see Lemma 1.5(a). From [12, Theorem 18.6(v)] we know that L is noetherian over \widehat{R} , so Lemma 1.3(b) implies that L is noetherian over R . \square

Mini-max and Matlis Reflexive Modules.

Definition 1.16. An R -module M is *mini-max* if there is a noetherian submodule $N \subseteq M$ such that M/N is artinian.

Definition 1.17. An R -module M is *Matlis reflexive* provided that the natural biduality map $\delta_M: M \rightarrow M^{\vee\vee}$, given by $\delta_M(x)(\psi) = \psi(x)$, is an isomorphism.

Fact 1.18. An R -module M is Matlis reflexive if and only if it is mini-max and $R/\text{Ann}_R(M)$ is complete; see [3, Theorem 12]. Thus, if M is mini-max over R , then $\widehat{R} \otimes_R M$ is Matlis reflexive over \widehat{R} .

Lemma 1.19. *If M is mini-max over R , then $\beta_i^R(M), \mu_R^i(M) < \infty$ for all $i \geq 0$.*

Proof. We show that $\mu_R^i(M) < \infty$ for all $i \geq 0$; then Remark 1.11(b) implies that $\beta_i^R(M) < \infty$ for all $i \geq 0$. The noetherian case is standard. If M is artinian, then we have $\mu^0 = \mu_R^0(M) < \infty$ by Lemma 1.14; since E^{μ^0} is artinian, an induction argument shows that $\mu_R^i(M) < \infty$ for all $i \geq 0$. One deduces the mini-max case from the artinian and noetherian cases, using a long exact sequence. \square

Lemma 1.20. *Let L be an R -module such that $R/\text{Ann}_R(L)$ is complete. The following conditions are equivalent:*

- (i) L is Matlis reflexive over R ;
- (ii) L is mini-max over R ;
- (iii) L is mini-max over \widehat{R} ; and
- (iv) L is Matlis reflexive over \widehat{R} .

Proof. The equivalences (i) \iff (ii) and (iii) \iff (iv) are from Fact 1.18. Note that conditions (iii) and (iv) make sense since L is an \widehat{R} -module; see Fact 1.2.

(ii) \implies (iii) Assume that L is mini-max over R , and fix a noetherian R -submodule $N \subseteq L$ such that L/N is artinian over R . As $R/\text{Ann}_R(N)$ is complete, Fact 1.2(d) and Lemma 1.3(a) imply that N is an \widehat{R} -submodule. Similarly, Lemmas 1.3(b) and 1.14 imply that N is noetherian over \widehat{R} , and L/N is an artinian over \widehat{R} . Thus L is mini-max over \widehat{R} .

(iii) \implies (ii) Assume that L is mini-max over \widehat{R} , and fix a noetherian \widehat{R} -submodule $L' \subseteq L$ such that L/L' is artinian over \widehat{R} . Lemmas 1.3(b) and 1.14 imply that L' is noetherian over R , and L/L' is artinian over R , so L is mini-max over R . \square

Lemma 1.21. *Let L be an R -module such that $\mathfrak{m}^t L = 0$ for some integer $t \geq 1$. Then the following conditions are equivalent:*

- (i) L is mini-max over R (equivalently, over \widehat{R});
- (ii) L is artinian over R (equivalently, over \widehat{R});
- (iii) L is noetherian over R (equivalently, over \widehat{R}); and
- (iv) L has finite length over R (equivalently, over \widehat{R}).

Proof. Lemma 1.20 shows that L is mini-max over R if and only if it is mini-max over \widehat{R} . Also, L is artinian (resp., noetherian or finite length) over R if and only if it is artinian (resp., noetherian or finite length) over \widehat{R} by Lemmas 1.14 and 1.3(b).

The equivalence of conditions (ii)–(iv) follows from an application of [6, Proposition 2.3.20] over the artinian ring R/\mathfrak{m}^t . The implication (ii) \implies (i) is evident. For

the implication (i) \implies (ii), assume that L is mini-max over R . Given a noetherian submodule $N \subseteq L$ such that L/N is artinian, the implication (iii) \implies (ii) shows that N is artinian; hence so is L . \square

Lemma 1.22. *The class of mini-max (resp., noetherian, artinian, finite length, or Matlis reflexive) R -modules is closed under submodules, quotients, and extensions.*

Proof. The noetherian, artinian, and finite length cases are standard, as is the Matlis reflexive case; see [6, p. 92, Exercise 2]. For the mini-max case, fix an exact sequence $0 \rightarrow L' \xrightarrow{f} L \xrightarrow{g} L'' \rightarrow 0$. Identify L' with $\text{Im}(f)$. Assume first that L is mini-max, and fix a noetherian submodule N such that L/N is artinian. Then $L' \cap N$ is noetherian, and the quotient $L'/(L' \cap N) \cong (L' + N)/N$ is artinian, since it is a submodule of L/N . Thus L' is mini-max. Also, $(N + L')/L'$ is noetherian and $[L/L']/[(N + L')/L'] \cong L/(N + L')$ is artinian, so $L'' \cong L/L'$ is mini-max.

Next, assume that L' and L'' are mini-max, and fix noetherian submodules $N' \subseteq L'$ and $N'' \subseteq L''$ such that L'/N' and L''/N'' are artinian. Let x_1, \dots, x_h be coset representatives in L of a generating set for N'' . Let $N = N' + Rx_1 + \dots + Rx_h$. Then N is noetherian and the following commutative diagram has exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N \cap L' & \longrightarrow & N & \longrightarrow & N'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' \longrightarrow 0. \end{array}$$

The sequence $0 \rightarrow L'/(N \cap L') \rightarrow L/N \rightarrow L''/N'' \rightarrow 0$ is exact by the Snake Lemma. The module $L'/(N \cap L')$ is artinian, being a quotient of L'/N' . Since the class of artinian modules is closed under extensions, the module L/N is artinian. It follows that L is mini-max. \square

The next two lemmas apply to the classes of modules from Lemma 1.22.

Lemma 1.23. *Let \mathcal{C} be a class R -modules that is closed under submodules, quotients, and extensions.*

- (a) *Given an exact sequence $L' \xrightarrow{f} L \xrightarrow{g} L''$, if $L', L'' \in \mathcal{C}$, then $L \in \mathcal{C}$.*
- (b) *Given an R -complex X and an integer i , if $X_i \in \mathcal{C}$, then $H_i(X) \in \mathcal{C}$.*
- (c) *Given a noetherian R -module N , if $L \in \mathcal{C}$, then $\text{Ext}_R^i(N, L), \text{Tor}_i^R(N, L) \in \mathcal{C}$.*

Proof. (a) Assume that $L', L'' \in \mathcal{C}$. By assumption, $\text{Im}(f), \text{Im}(g) \in \mathcal{C}$. Using the exact sequence $0 \rightarrow \text{Im}(f) \rightarrow L \rightarrow \text{Im}(g) \rightarrow 0$, we conclude that L is in \mathcal{C} .

(b) The module $H_i(X)$ is a subquotient of X_i , so it is in \mathcal{C} by assumption.

(c) If F is a minimal free resolution of N , then the modules in the complexes $\text{Hom}_R(F, L)$ and $F \otimes_R L$ are in \mathcal{C} , so their homologies are in \mathcal{C} by part (b). \square

Lemma 1.24. *Let $R \rightarrow S$ be a local ring homomorphism, and let \mathcal{C} be a class of S -modules that is closed under submodules, quotients, and extensions. Fix an S -module L , an R -module L' , an R -submodule $L'' \subseteq L'$, and an index $i \geq 0$.*

- (a) *If $\text{Ext}_R^i(L, L''), \text{Ext}_R^i(L, L'/L'') \in \mathcal{C}$, then $\text{Ext}_R^i(L, L') \in \mathcal{C}$.*
- (b) *If $\text{Ext}_R^i(L'', L), \text{Ext}_R^i(L'/L'', L) \in \mathcal{C}$, then $\text{Ext}_R^i(L', L) \in \mathcal{C}$.*
- (c) *If $\text{Tor}_i^R(L, L''), \text{Tor}_i^R(L, L'/L'') \in \mathcal{C}$, then $\text{Tor}_i^R(L, L') \in \mathcal{C}$.*

Proof. We prove part (a); the other parts are proved similarly. Apply $\text{Ext}_R^i(L, -)$ to the exact sequence $0 \rightarrow L'' \rightarrow L' \rightarrow L'/L'' \rightarrow 0$ to obtain the next exact sequence:

$$\text{Ext}_R^i(L, L'') \rightarrow \text{Ext}_R^i(L, L') \rightarrow \text{Ext}_R^i(L, L'/L'').$$

Since L is an S -module, the maps in this sequence are S -module homomorphisms. Now, apply Lemma 1.23(a). \square

2. PROPERTIES OF $\text{Ext}_R^i(M, -)$

This section documents properties of the functors $\text{Ext}_R^i(M, -)$ where M is a mini-max R -module.

Noetherianness of $\text{Ext}_R^i(A, L)$.

Lemma 2.1. *Let A and L be R -modules such that A is artinian and L is \mathfrak{m} -torsion.*

- (a) *Then $\text{Hom}_R(L, A) = \text{Hom}_{\widehat{R}}(L, A) \cong \text{Hom}_{\widehat{R}}(A^\vee, L^\vee)$.*
- (b) *If L is artinian, then $\text{Hom}_R(L, A)$ is a noetherian \widehat{R} -module.*

Proof. (a) The first equality is from Lemma 1.5(a). For the second equality, the fact that A is Matlis reflexive over \widehat{R} explains the first step below:

$$\text{Hom}_{\widehat{R}}(L, A) \cong \text{Hom}_{\widehat{R}}(L, A^{vv}) \cong \text{Hom}_{\widehat{R}}(A^v, L^v) \cong \text{Hom}_{\widehat{R}}(A^\vee, L^\vee)$$

where $(-)^v = \text{Hom}_{\widehat{R}}(-, E)$. The second step follows from Hom-tensor adjointness, and the third step is from Lemma 1.5(a).

(b) If L is artinian, then L^\vee and A^\vee are noetherian over \widehat{R} , so $\text{Hom}_{\widehat{R}}(A^\vee, L^\vee)$ is also noetherian over \widehat{R} . \square

The next result contains part of Theorem 2 from the introduction. When R is not complete, the example $\text{Hom}_R(E, E) \cong \widehat{R}$ shows that $\text{Ext}_R^i(A, L)$ is not necessarily noetherian or artinian over R .

Theorem 2.2. *Let A and L be R -modules such that A is artinian. For each index $i \geq 0$ such that $\mu_R^i(L) < \infty$, the module $\text{Ext}_R^i(A, L)$ is a noetherian \widehat{R} -module.*

Proof. Let J be a minimal R -injective resolution of L . Remark 1.11(a) implies that $\Gamma_{\mathfrak{m}}(J)^i \cong E^{\mu_R^i(L)}$. Lemma 1.5(b) explains the first isomorphism below:

$$\text{Hom}_R(A, J)^i \cong \text{Hom}_R(A, \Gamma_{\mathfrak{m}}(J)^i) \cong \text{Hom}_R(A, E)^{\mu_R^i(L)}.$$

Lemma 2.1 implies that these are noetherian \widehat{R} -modules. The differentials in the complex $\text{Hom}_R(A, \Gamma_{\mathfrak{m}}(J))$ are \widehat{R} -linear because A is an \widehat{R} -module. Thus, the subquotient $\text{Ext}_R^i(A, L)$ is a noetherian \widehat{R} -module. \square

Corollary 2.3. *Let A and M be R -modules such that A is artinian and M is mini-max. For each index $i \geq 0$, the module $\text{Ext}_R^i(A, M)$ is a noetherian \widehat{R} -module.*

Proof. Apply Theorem 2.2 and Lemma 1.19. \square

The next result contains part of Theorem 4 from the introduction.

Corollary 2.4. *Let A and L be R -modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is complete and A is artinian. For each index $i \geq 0$ such that $\mu_R^i(L) < \infty$, the module $\text{Ext}_R^i(A, L)$ is noetherian and Matlis reflexive over R and \widehat{R} .*

Proof. Theorem 2.2 shows that $\text{Ext}_R^i(A, L)$ is noetherian over \widehat{R} ; so, it is Matlis reflexive over \widehat{R} . As $\text{Ann}_R(A) + \text{Ann}_R(L) \subseteq \text{Ann}_R(\text{Ext}_R^i(A, L))$, Lemmas 1.3(b) and 1.20 imply that $\text{Ext}_R^i(A, L)$ is noetherian and Matlis reflexive over R . \square

Corollary 2.5. *Let A and L be R -modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is artinian and A is artinian. Given an index $i \geq 0$ such that $\mu_R^i(L) < \infty$, one has $\text{len}_R(\text{Ext}_R^i(A, L)) < \infty$.*

Proof. Apply Theorem 2.2 and Lemma 1.21. \square

Matlis Reflexivity of $\text{Ext}_R^i(M, M')$.

Theorem 2.6. *Let A and M be R -modules such that A is artinian and M is mini-max. For each $i \geq 0$, the module $\text{Ext}_R^i(M, A)$ is Matlis reflexive over \widehat{R} .*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. Since A is artinian, it is an \widehat{R} -module. Corollary 2.3 implies that $\text{Ext}_R^i(M/N, A)$ is a noetherian \widehat{R} -module. As $\text{Ext}_R^i(N, A)$ is artinian, Lemma 1.24(b) says that $\text{Ext}_R^i(M, A)$ is a mini-max \widehat{R} -module and hence is Matlis reflexive over \widehat{R} by Fact 1.18. \square

Theorem 2.7. *Let M and N' be R -modules such that M is mini-max and N' is noetherian. Fix an index $i \geq 0$. If $R/(\text{Ann}_R(M) + \text{Ann}_R(N'))$ is complete, then $\text{Ext}_R^i(M, N')$ is noetherian and Matlis reflexive over R and over \widehat{R} .*

Proof. Fix a noetherian submodule $N \subseteq M$ such that M/N is artinian. If the ring $R/(\text{Ann}_R(M) + \text{Ann}_R(N'))$ is complete, then so is $R/(\text{Ann}_R(M/N) + \text{Ann}_R(N'))$. Corollary 2.4 implies that $\text{Ext}_R^i(M/N, N')$ is noetherian over R . Since $\text{Ext}_R^i(N, N')$ is noetherian over R , Lemma 1.24(b) implies that $\text{Ext}_R^i(M, N')$ is noetherian over R . As $R/(\text{Ann}_R(\text{Ext}_R^i(M, N')))$ is complete, Fact 1.18 implies that $\text{Ext}_R^i(M, N')$ is also Matlis reflexive over R . Thus $\text{Ext}_R^i(M, N')$ is noetherian and Matlis reflexive over \widehat{R} by Lemmas 1.3(b) and 1.20. \square

Theorem 2.8. *Let M and M' be mini-max R -modules, and fix an index $i \geq 0$.*

- (a) *If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete, then $\text{Ext}_R^i(M, M')$ is Matlis reflexive over R and \widehat{R} .*
- (b) *If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is artinian, then $\text{Ext}_R^i(M, M')$ has finite length.*

Proof. Fix a noetherian submodule $N' \subseteq M'$ such that M'/N' is artinian.

(a) Assume that $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete. Theorem 2.7 implies that the module $\text{Ext}_R^i(M, N')$ is Matlis reflexive over R . Theorem 2.6 shows that $\text{Ext}_R^i(M, M'/N')$ is Matlis reflexive over \widehat{R} ; hence, it is Matlis reflexive over R by Lemma 1.20. Thus, Lemmas 1.24(a) and 1.20 imply that $\text{Ext}_R^i(M, M')$ is Matlis reflexive over R and \widehat{R} .

(b) This follows from part (a), because of Fact 1.18 and Lemma 1.21. \square

A special case of the next result can be found in [2, Theorem 3].

Corollary 2.9. *Let M and M' be R -modules such that M is mini-max and M' is Matlis reflexive. For each index $i \geq 0$, the modules $\text{Ext}_R^i(M, M')$ and $\text{Ext}_R^i(M', M)$ are Matlis reflexive over R and \widehat{R} .*

Proof. Apply Theorem 2.8(a) and Fact 1.18. \square

Length Bounds for $\text{Hom}_R(A, L)$.

Lemma 2.10. *Let A and L be R -modules such that A is artinian and $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$ for some $n \geq 1$. Fix an index $t \geq 0$ such that $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$, and let s be an integer such that $s \geq \min(n, t)$. Then*

$$\text{Hom}_R(A, L) \cong \text{Hom}_R(A/\mathfrak{m}^s A, L) \cong \text{Hom}_R(A/\mathfrak{m}^s A, (0 :_L \mathfrak{m}^s)).$$

Proof. Given any map $\psi \in \text{Hom}_R(A/\mathfrak{m}^s A, L)$, the image of ψ is annihilated by \mathfrak{m}^s . That is, $\text{Im}(\psi) \subseteq (0 :_L \mathfrak{m}^s)$; hence $\text{Hom}_R(A/\mathfrak{m}^s A, L) \cong \text{Hom}_R(A/\mathfrak{m}^s A, (0 :_L \mathfrak{m}^s))$. In the next sequence, the first and third isomorphisms are from Lemma 1.5(b):

$$\text{Hom}_R(A, L) \cong \text{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) \cong \text{Hom}_R(A/\mathfrak{m}^s A, \Gamma_{\mathfrak{m}}(L)) \cong \text{Hom}_R(A/\mathfrak{m}^s A, L).$$

For the second isomorphism, we argue by cases. If $s \geq n$, then we have $\mathfrak{m}^s \Gamma_{\mathfrak{m}}(L) = 0$ because $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$, and the isomorphism is evident. If $s < n$, then we have $n > s \geq t$, so $\mathfrak{m}^t A = \mathfrak{m}^s A = \mathfrak{m}^n A$ since $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$; it follows that $\text{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) \cong \text{Hom}_R(A/\mathfrak{m}^n A, \Gamma_{\mathfrak{m}}(L)) \cong \text{Hom}_R(A/\mathfrak{m}^s A, \Gamma_{\mathfrak{m}}(L))$. \square

For the next result, the example $\text{Hom}_R(E, E) \cong \widehat{R}$ shows that the condition $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$ is necessary.

Theorem 2.11. *Let A and L be R -modules such that A is artinian and $\mathfrak{m}^n \Gamma_{\mathfrak{m}}(L) = 0$ for some $n \geq 1$. Fix an index $t \geq 0$ such that $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$, and let s be an integer such that $s \geq \min(n, t)$. Then there is an inequality*

$$\text{len}_R(\text{Hom}_R(A, L)) \leq \beta_0^R(A) \text{len}_R(0 :_L \mathfrak{m}^s).$$

Here, we use the convention $0 \cdot \infty = 0$.

Proof. We deal with the degenerate case first. If $\beta_0^R(A) = 0$, then $A/\mathfrak{m}A = 0$, so

$$\text{Hom}_R(A, L) \cong \text{Hom}_R(A/\mathfrak{m}A, L) = \text{Hom}_R(0, L) = 0$$

by Lemma 2.10. So, we assume for the rest of the proof that $\beta_0^R(A) \neq 0$. We also assume without loss of generality that $\text{len}_R(0 :_L \mathfrak{m}^s) < \infty$.

Lemma 2.10 explains the first step in the following sequence:

$$\begin{aligned} \text{len}_R(\text{Hom}_R(A, L)) &= \text{len}_R(\text{Hom}_R(A/\mathfrak{m}^s A, (0 :_L \mathfrak{m}^s))) \\ &\leq \beta_0^R(A/\mathfrak{m}^s A) \text{len}_R(0 :_L \mathfrak{m}^s) \\ &= \beta_0^R(A) \text{len}_R(0 :_L \mathfrak{m}^s). \end{aligned}$$

The second step can be proved by induction on $\beta_0^R(A/\mathfrak{m}^s A)$ and $\text{len}_R(0 :_L \mathfrak{m}^s)$. \square

The next result gives part of Theorem 1 from the introduction. Example 6.3 shows that one should not expect to have $\text{len}_R(\text{Ext}_R^i(A, N)) < \infty$ when $i \geq 1$.

Corollary 2.12. *If A and N are R -modules such that A is artinian and N is noetherian, then $\text{len}_R(\text{Hom}_R(A, N)) < \infty$.*

Proof. Apply Theorem 2.11 and Lemma 1.19. \square

3. PROPERTIES OF $\text{Tor}_i^R(M, -)$

This section focuses on properties of the functors $\text{Tor}_i^R(M, -)$ where M is a mini-max R -module.

Artinianness of $\text{Tor}_i^R(\mathbf{A}, \mathbf{L})$.

The next result contains part of Theorem 2 from the introduction. Recall that a module is artinian over R if and only if it is artinian over \widehat{R} ; see Lemma 1.14.

Theorem 3.1. *Let A and L be R -modules such that A is artinian. For each index $i \geq 0$ such that $\beta_i^R(L) < \infty$, the module $\text{Tor}_i^R(A, L)$ is artinian.*

Proof. Lemma 1.13(b) implies that $\mu_R^i(L^\vee) = \beta_i^R(L) < \infty$. By Remark 1.9, we have $\text{Ext}_R^i(A, L^\vee) \cong \text{Tor}_i^R(A, L)^\vee$. Thus, $\text{Tor}_i^R(A, L)^\vee$ is a noetherian \widehat{R} -module by Theorem 2.2, and we conclude that $\text{Tor}_i^R(A, L)$ is artinian by Lemma 1.15(b). \square

For the next result, the example $E \otimes_R R \cong E$ shows that $\text{Tor}_i^R(A, L)$ is not necessarily noetherian over R or \widehat{R} .

Corollary 3.2. *Let A and M be R -modules such that A is artinian and M mini-max. For each index $i \geq 0$, the module $\text{Tor}_i^R(A, M)$ is artinian.*

Proof. Apply Theorem 3.1 and Lemma 1.19. \square

The proofs of the next two results are similar to those of Corollaries 2.4 and 2.5. The first result contains part of Theorem 4 from the introduction.

Corollary 3.3. *Let A and L be R -modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is complete and A is artinian. For each index $i \geq 0$ such that $\beta_i^R(L) < \infty$, the module $\text{Tor}_i^R(A, L)$ is artinian and Matlis reflexive over R and \widehat{R} .*

Corollary 3.4. *Let A and L be R -modules such that $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is artinian and A is artinian. Given an index $i \geq 0$ such that $\beta_i^R(L) < \infty$, one has $\text{len}_R(\text{Tor}_i^R(A, L)) < \infty$.*

 $\text{Tor}_i^R(\mathbf{M}, \mathbf{M}')$ is Mini-max.

Theorem 3.5. *Let M and M' be mini-max R -modules, and fix an index $i \geq 0$.*

- (a) *The R -module $\text{Tor}_i^R(M, M')$ is mini-max over R .*
- (b) *If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete, then $\text{Tor}_i^R(M, M')$ is Matlis reflexive over R and \widehat{R} .*
- (c) *If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is artinian, then $\text{Tor}_i^R(M, M')$ has finite length.*

Proof. (a) Choose a noetherian submodule $N \subseteq M$ such that M/N is artinian. Lemmas 1.22 and 1.23(c) say that $\text{Tor}_i^R(N, M')$ is mini-max. Corollary 3.2 implies that $\text{Tor}_i^R(M/N, M')$ is mini-max, so $\text{Tor}_i^R(M, M')$ is mini-max by Lemma 1.24(c).

Parts (b) and (c) now follow from Lemmas 1.20 and 1.21. \square

A special case of the next result is contained in [2, Theorem 3].

Corollary 3.6. *Let M and M' be R -modules such that M is mini-max and M' is Matlis reflexive. For each index $i \geq 0$, the module $\text{Tor}_i^R(M, M')$ is Matlis reflexive over R and \widehat{R} .*

Proof. Apply Theorem 3.5(b) and Fact 1.18. \square

Length Bounds for $A \otimes_R L$.

Lemma 3.7. *Let A be an artinian module, and let \mathfrak{a} be a proper ideal of R . Fix an integer $t \geq 0$ such that $\mathfrak{a}^t A = \mathfrak{a}^{t+1} A$. Given an \mathfrak{a} -torsion R -module L , one has*

$$A \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L).$$

Proof. The isomorphism $(A/\mathfrak{a}^t A) \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L)$ is from the following:

$$\begin{aligned} (A/\mathfrak{a}^t A) \otimes_R L &\cong [(A/\mathfrak{a}^t A) \otimes_R (R/\mathfrak{a}^t)] \otimes_R L \\ &\cong (A/\mathfrak{a}^t A) \otimes_R [(R/\mathfrak{a}^t) \otimes_R L] \\ &\cong (A/\mathfrak{a}^t A) \otimes_R (L/\mathfrak{a}^t L). \end{aligned}$$

For the isomorphism $A \otimes_R L \cong (A/\mathfrak{a}^t A) \otimes_R L$, consider the exact sequence:

$$0 \rightarrow \mathfrak{a}^t A \rightarrow A \rightarrow A/\mathfrak{a}^t A \rightarrow 0.$$

The exact sequence induced by $- \otimes_R L$ has the form

$$(3.7.1) \quad (\mathfrak{a}^t A) \otimes_R L \rightarrow A \otimes_R L \rightarrow (A/\mathfrak{a}^t A) \otimes_R L \rightarrow 0.$$

The fact that L is \mathfrak{a} -torsion and $\mathfrak{a}^t A = \mathfrak{a}^{t+i} A$ for all $i \geq 1$ implies that $(\mathfrak{a}^t A) \otimes_R L = 0$, so the sequence (3.7.1) yields the desired isomorphism. \square

The example $E \otimes_R R \cong R$ shows that the \mathfrak{m} -torsion assumption on L is necessary in the next result.

Theorem 3.8. *Let A be an artinian R -module, and let L be an \mathfrak{m} -torsion R -module. Fix an integer $t \geq 0$ such that $\mathfrak{m}^t A = \mathfrak{m}^{t+1} A$. Then there are inequalities*

$$(3.8.1) \quad \text{len}_R(A \otimes_R L) \leq \text{len}_R(A/\mathfrak{m}^t A) \beta_0^R(L)$$

$$(3.8.2) \quad \text{len}_R(A \otimes_R L) \leq \beta_0^R(A) \text{len}_R(L/\mathfrak{m}^t L).$$

Here we use the convention $0 \cdot \infty = 0$.

Proof. From Lemma 3.7 we have

$$(3.8.3) \quad A \otimes_R L \cong (A/\mathfrak{m}^t A) \otimes_R (L/\mathfrak{m}^t L).$$

Lemmas 1.19 and 1.21 imply that $\text{len}_R(A/\mathfrak{m}^t A) < \infty$ and $\beta_0^R(A) < \infty$.

For the degenerate cases, first note that $\text{len}_R(A/\mathfrak{m}^t A) = 0$ if and only if $\beta_0^R(A) = 0$. When $\text{len}_R(A/\mathfrak{m}^t A) = 0$, the isomorphism (3.8.3) implies that $A \otimes_R L = 0$; hence the desired inequalities. Thus, we assume without loss of generality that $1 \leq \beta_0^R(A) \leq \text{len}_R(A/\mathfrak{m}^t A)$. Further, we assume that $\beta_0^R(L) < \infty$.

The isomorphism (3.8.3) provides the first step in the next sequence:

$$\text{len}_R(A \otimes_R L) = \text{len}_R((A/\mathfrak{m}^t A) \otimes_R (L/\mathfrak{m}^t L)) \leq \text{len}_R(A/\mathfrak{m}^t A) \beta_0^R(L).$$

The second step in this sequence can be verified by induction on $\text{len}_R(A/\mathfrak{m}^t A)$ and $\beta_0^R(L)$. This explains the inequality (3.8.1), and (3.8.2) is verified similarly. \square

The next result contains part of Theorem 1 from the introduction. Example 6.4 shows that one should not expect to have $\text{len}_R(\text{Tor}_i^R(A, A')) < \infty$ when $i \geq 1$.

Corollary 3.9. *If A and A' are artinian R -modules, then $\text{len}_R(A \otimes_R A') < \infty$.*

Proof. Apply Theorem 3.8 and Lemmas 1.19 and 1.21. \square

4. THE MATLIS DUAL OF $\text{Ext}_R^i(L, L')$

This section contains the proof of Theorem 5 from the introduction; see Corollary 4.11. Most of the section is devoted to technical results for use in the proof.

Lemma 4.1. *Let L be an R -module. If I is an R -injective resolution of L , and J is an \widehat{R} -injective resolution of $\widehat{R} \otimes_R L$, then there is a homotopy equivalence $\Gamma_{\mathfrak{m}}(I) \xrightarrow{\sim} \Gamma_{\mathfrak{m}}(J) = \Gamma_{\mathfrak{m}\widehat{R}}(J)$.*

Proof. Each injective \widehat{R} -module J' is injective over R ; this follows from the isomorphism $\text{Hom}_R(-, J') \cong \text{Hom}_R(-, \text{Hom}_{\widehat{R}}(\widehat{R}, J')) \cong \text{Hom}_{\widehat{R}}(\widehat{R} \otimes_R -, J')$ since \widehat{R} is flat over R . Hence, there is a lift $f: I \rightarrow J$ of the natural map $\xi: L \rightarrow \widehat{R} \otimes_R L$. This lift is a chain map of R -complexes.

We show that the induced map $\Gamma_{\mathfrak{m}}(f): \Gamma_{\mathfrak{m}}(I) \rightarrow \Gamma_{\mathfrak{m}}(J) = \Gamma_{\mathfrak{m}\widehat{R}}(J)$ is a homotopy equivalence. As $\Gamma_{\mathfrak{m}}(I)$ and $\Gamma_{\mathfrak{m}}(J)$ are bounded above complexes of injective R -modules, it suffices to show that $\Gamma_{\mathfrak{m}}(f)$ induces an isomorphism on homology in each degree. The induced map on homology is compatible with the following sequence:

$$H^i(\Gamma_{\mathfrak{m}}(I)) \cong H_{\mathfrak{m}}^i(L) \xrightarrow[\cong]{H_{\mathfrak{m}}^i(\xi)} H_{\mathfrak{m}}^i(\widehat{R} \otimes_R L) \cong H^i(\Gamma_{\mathfrak{m}}(J)).$$

The map $H_{\mathfrak{m}}^i(\xi): H_{\mathfrak{m}}^i(L) \rightarrow H_{\mathfrak{m}}^i(\widehat{R} \otimes_R L)$ is an isomorphism (see the proof of [5, Proposition 3.5.4(d)]) so we have the desired homotopy equivalence. \square

Lemma 4.2. *Let L and L' be R -modules such that L is \mathfrak{m} -torsion. Then for each index $i \geq 0$, there are \widehat{R} -module isomorphisms*

$$\text{Ext}_R^i(L, L') \cong \text{Ext}_R^i(L, \widehat{R} \otimes_R L') \cong \text{Ext}_{\widehat{R}}^i(L, \widehat{R} \otimes_R L').$$

Proof. Let I be an R -injective resolution of L' , and let J be an \widehat{R} -injective resolution of $\widehat{R} \otimes_R L'$. Because L is \mathfrak{m} -torsion, Lemma 1.5(b) explains the first, third and sixth steps in the next display:

$$\begin{aligned} \text{Hom}_R(L, I) &\cong \text{Hom}_R(L, \Gamma_{\mathfrak{m}}(I)) \sim \text{Hom}_R(L, \Gamma_{\mathfrak{m}}(J)) \cong \text{Hom}_R(L, J) \\ \text{Hom}_R(L, \Gamma_{\mathfrak{m}}(J)) &= \text{Hom}_R(L, \Gamma_{\mathfrak{m}\widehat{R}}(J)) = \text{Hom}_{\widehat{R}}(L, \Gamma_{\mathfrak{m}\widehat{R}}(J)) \cong \text{Hom}_{\widehat{R}}(L, J). \end{aligned}$$

The homotopy equivalence in the second step is from Lemma 4.1. The fifth step is from Lemma 1.5(a). Since L is \mathfrak{m} -torsion, it is an \widehat{R} -module, so the isomorphisms and the homotopy equivalence in this sequence are \widehat{R} -linear. In particular, the complexes $\text{Hom}_R(L, I)$ and $\text{Hom}_R(L, J)$ and $\text{Hom}_{\widehat{R}}(L, J)$ have isomorphic cohomology over \widehat{R} , so one has the desired isomorphisms. \square

The next result contains Theorem 3 from the introduction. It shows, for instance, that given artinian R -modules A and A' , there are noetherian \widehat{R} -modules N and N' such that $\text{Ext}_R^i(A, A') \cong \text{Ext}_{\widehat{R}}^i(N, N')$; thus, it provides an alternate proof of Corollary 2.3.

Theorem 4.3. *Let A and M be R -modules such that A is artinian and M is mini-max. Then for each index $i \geq 0$, we have $\text{Ext}_R^i(A, M) \cong \text{Ext}_{\widehat{R}}^i(M^\vee, A^\vee)$.*

Proof. Case 1: R is complete. Let F be a free resolution of A . It follows that each F_i is flat, so the complex F^\vee is an injective resolution of A^\vee ; see [6, Theorem 3.2.9].

We obtain the isomorphism $\text{Ext}_R^i(A, M) \cong \text{Ext}_R^i(M^\vee, A^\vee)$ by taking cohomology in the next sequence:

$$\text{Hom}_R(F, M) \cong \text{Hom}_R(F, M^{\vee\vee}) \cong \text{Hom}_R(M^\vee, F^\vee).$$

The first step follows from the fact that M is Matlis reflexive; see Fact 1.18. The second step is from Hom-tensor adjointness

Case 2: the general case. The first step below is from Lemma 4.2:

$$\text{Ext}_R^i(A, M) \cong \text{Ext}_{\widehat{R}}^i(A, \widehat{R} \otimes_R M) \cong \text{Ext}_{\widehat{R}}^i((\widehat{R} \otimes_R M)^\vee, A^\vee) \cong \text{Ext}_{\widehat{R}}^i(M^\vee, A^\vee).$$

Here $(-)^\vee = \text{Hom}_{\widehat{R}}(-, E)$. Since M is mini-max, it follows that $\widehat{R} \otimes_R M$ is mini-max over \widehat{R} . Thus, the second step is from Case 1. For the third step use Hom-tensor adjointness and Lemma 1.5(a) to see that $(\widehat{R} \otimes_R M)^\vee \cong M^\vee$ and $A^\vee \cong A^\vee$. \square

Fact 4.4. Let L and L' be R -modules, and fix an index $i \geq 0$. Then the following diagram commutes, where the unlabeled isomorphism is from Remark 1.9:

$$\begin{array}{ccc} \text{Ext}_R^i(L', L) & \xrightarrow{\delta_{\text{Ext}_R^i(L', L)}} & \text{Ext}_R^i(L', L)^{\vee\vee} \\ \text{Ext}_R^i(L', \delta_L) \downarrow & & \downarrow (\Theta_{L', L}^i)^\vee \\ \text{Ext}_R^i(L', L^{\vee\vee}) & \xrightarrow{\cong} & \text{Tor}_i^R(L', L^\vee)^\vee. \end{array}$$

Lemma 4.5. Let L be an R -module, and fix an index $i \geq 0$. If $\mu_R^i(L) < \infty$, then the map $\text{Ext}_R^i(k, \delta_L): \text{Ext}_R^i(k, L) \rightarrow \text{Ext}_R^i(k, L^{\vee\vee})$ is an isomorphism.

Proof. The assumption $\mu_R^i(L) < \infty$ says that $\text{Ext}_R^i(k, L)$ is a finite dimensional k -vector space, so it is Matlis reflexive over R ; that is, the map

$$\delta_{\text{Ext}_R^i(k, L)}: \text{Ext}_R^i(k, L) \rightarrow \text{Ext}_R^i(k, L)^{\vee\vee}$$

is an isomorphism. Since k is finitely generated, Remark 1.9 implies that

$$\Theta_{kL}^i: \text{Tor}_i^R(k, L^\vee) \rightarrow \text{Ext}_R^i(k, L)^\vee$$

is an isomorphism. Hence $(\Theta_{kL}^i)^\vee$ is also an isomorphism. Using Fact 4.4 with $L' = k$, we conclude that $\text{Ext}_R^i(k, \delta_L)$ is an isomorphism, as desired. \square

Lemma 4.6. Let A and L be R -modules such that A is artinian. Fix an index $i \geq 0$ such that $\mu_R^{i-1}(L)$, $\mu_R^i(L)$ and $\mu_R^{i+1}(L)$ are finite. Then the map

$$\text{Ext}_R^i(A, \delta_L): \text{Ext}_R^i(A, L) \rightarrow \text{Ext}_R^i(A, L^{\vee\vee})$$

is an isomorphism.

Proof. Lemma 4.5 implies that for $t = i - 1, i, i + 1$ the maps

$$\text{Ext}_R^t(k, \delta_L): \text{Ext}_R^t(k, L) \rightarrow \text{Ext}_R^t(k, L^{\vee\vee})$$

are isomorphisms. As the biduality map δ_L is injective, we have an exact sequence

$$(4.6.1) \quad 0 \rightarrow L \xrightarrow{\delta_L} L^{\vee\vee} \rightarrow \text{Coker } \delta_L \rightarrow 0.$$

Using the long exact sequence associated to $\text{Ext}_R(k, -)$, we conclude that for $t = i - 1, i$ we have $\text{Ext}_R^t(k, \text{Coker } \delta_L) = 0$. In other words, we have $\mu_R^t(\text{Coker } \delta_L) = 0$.

Let J be a minimal injective resolution of $\text{Coker } \delta_L$. The previous paragraph shows that for $t = i - 1, i$ the module J^t does not have E as a summand by Remark 1.11(a). That is, we have $\Gamma_{\mathfrak{m}}(J^t) = 0$, so Lemma 1.5(b) implies that

$$\text{Hom}_R(A, J^t) \cong \text{Hom}_R(A, \Gamma_{\mathfrak{m}}(J^t)) = 0.$$

It follows that $\text{Ext}_R^t(A, \text{Coker}(\delta_L)) = 0$ for $t = i - 1, i$. From the long exact sequence associated to $\text{Ext}_R(A, -)$ with respect to (4.6.1), it follows that $\text{Ext}_R^i(A, \delta_L)$ is an isomorphism, as desired. \square

We are now ready to tackle the main results of this section.

Theorem 4.7. *Let A and L be R -modules such that A is artinian. Fix an index $i \geq 0$ such that $\mu_R^{i-1}(L)$, $\mu_R^i(L)$ and $\mu_R^{i+1}(L)$ are finite.*

- (a) *There is an R -module isomorphism $\text{Ext}_R^i(A, L)^v \cong \text{Tor}_i^R(A, L^\vee)$ where $(-)^v = \text{Hom}_{\widehat{R}}(-, E)$.*
- (b) *If $R/(\text{Ann}_R(A) + \text{Ann}_R(L))$ is complete, then Θ_{AL}^i provides an isomorphism $\text{Tor}_i^R(A, L^\vee) \cong \text{Ext}_R^i(A, L)^v$.*

Proof. (b) Corollary 2.4 and Lemma 4.6 show that the maps

$$\begin{aligned} \delta_{\text{Ext}_R^i(A, L)}: \text{Ext}_R^i(A, L) &\rightarrow \text{Ext}_R^i(A, L)^{\vee\vee} \\ \text{Ext}_R^i(A, \delta_L): \text{Ext}_R^i(A, L) &\rightarrow \text{Ext}_R^i(A, L^{\vee\vee}) \end{aligned}$$

are isomorphisms. Fact 4.4 implies that $(\Theta_{AL}^i)^\vee$ is an isomorphism, so we conclude that Θ_{AL}^i is also an isomorphism.

- (a) Lemma 4.2 explains the first step in the next sequence:

$$\begin{aligned} \text{Ext}_R^i(A, L)^v &\cong \text{Ext}_{\widehat{R}}^i(A, \widehat{R} \otimes_R L)^v \\ &\cong \text{Tor}_{\widehat{R}}^i(A, (\widehat{R} \otimes_R L)^v) \\ &\cong \text{Tor}_i^R(A, (\widehat{R} \otimes_R L)^v) \\ &\cong \text{Tor}_i^R(A, L^\vee). \end{aligned}$$

The second step is from part (b), as \widehat{R} is complete and $\mu_{\widehat{R}}^t(\widehat{R} \otimes_R L) = \mu_R^t(L) < \infty$ for $t = i - 1, i, i + 1$. The fourth step is from Hom-tensor adjointness. For the third step, let P be a projective resolution of A over R . Since \widehat{R} is flat over R , the complex $\widehat{R} \otimes_R P$ is a projective resolution of $\widehat{R} \otimes_R A \cong A$ over \widehat{R} ; see Lemma 1.4(a). Thus, the third step follows from the isomorphism $(\widehat{R} \otimes_R P) \otimes_{\widehat{R}} (\widehat{R} \otimes_R L)^v \cong P \otimes_R (\widehat{R} \otimes_R L)^v$. \square

Question 4.8. Do the conclusions of Lemma 4.6 and Theorem 4.7 hold when one only assumes that $\mu_R^i(L)$ is finite?

Corollary 4.9. *Let A and M be R -modules such that A is artinian and M is mini-max. For each index $i \geq 0$, one has $\text{Ext}_R^i(A, M)^v \cong \text{Tor}_i^R(A, M^\vee)$, where $(-)^v = \text{Hom}_{\widehat{R}}(-, E)$.*

Proof. Apply Theorem 4.7(a) and Lemma 1.19. \square

Theorem 4.10. *Let M and M' be mini-max R -modules, and fix an index $i \geq 0$. If $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete, then $\Theta_{MM'}^i$ is an isomorphism, so*

$$\text{Ext}_R^i(M, M')^v = \text{Ext}_R^i(M, M')^\vee \cong \text{Tor}_i^R(M, M'^\vee)$$

where $(-)^v = \text{Hom}_{\widehat{R}}(-, E)$.

Proof. Theorem 2.8(a) implies that $\text{Ext}_R^i(M, M')$ is Matlis reflexive over R , so Lemma 1.5(a) and Fact 1.18 imply that $\text{Ext}_R^i(M, M')^v = \text{Ext}_R^i(M, M')^\vee$. Thus, it remains to show that $\Theta_{MM'}^i$ is an isomorphism.

Case 1: M is noetherian. In the next sequence, the first and last steps are from Hom-tensor adjointness. The second step is standard since M is noetherian:

$$\begin{aligned} \text{Ext}_R^i(M, M')^\vee &\cong (\widehat{R} \otimes_R \text{Ext}_R^i(M, M'))^v \\ &\cong \text{Ext}_{\widehat{R}}^i(\widehat{R} \otimes_R M, \widehat{R} \otimes_R M')^v \\ &\cong \text{Tor}_{\widehat{R}}^i(\widehat{R} \otimes_R M, (\widehat{R} \otimes_R M')^v) \\ &\cong \text{Tor}_i^R(M, (\widehat{R} \otimes_R M')^v) \\ &\cong \text{Tor}_i^R(M, M'^\vee). \end{aligned}$$

Since M and M' are mini-max over R , the modules $\widehat{R} \otimes_R M$ and $\widehat{R} \otimes_R M'$ are Matlis reflexive over \widehat{R} ; see Fact 1.18. Thus [1, Theorem 4(c)] explains the third step. The fourth step is from the fact that \widehat{R} is flat over R . Since these isomorphisms are compatible with $\Theta_{MM'}^i$, it follows that $\Theta_{MM'}^i$ is an isomorphism.

Case 2: the general case. Since M is mini-max over R , there is an exact sequence of R -modules homomorphisms $0 \rightarrow N \rightarrow M \rightarrow A \rightarrow 0$ such that N is noetherian and A is artinian. The long exact sequences associated to $\text{Tor}^R(-, M'^\vee)$ and $\text{Ext}_R(-, M')$ fit into the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Tor}_i^R(N, M'^\vee) & \longrightarrow & \text{Tor}_i^R(M, M'^\vee) & \longrightarrow & \text{Tor}_i^R(A, M'^\vee) \longrightarrow \cdots \\ & & \downarrow \Theta_{NM'}^i & & \downarrow \Theta_{MM'}^i & & \downarrow \Theta_{AM'}^i \\ \cdots & \longrightarrow & \text{Ext}_R^i(N, M')^\vee & \longrightarrow & \text{Ext}_R^i(M, M')^\vee & \longrightarrow & \text{Ext}_R^i(A, M')^\vee \longrightarrow \cdots \end{array}$$

Case 1 shows that $\Theta_{NM'}^i$ and $\Theta_{NM'}^{i-1}$ are isomorphisms. Theorem 4.7(b) implies that $\Theta_{AM'}^i$ and $\Theta_{AM'}^{i+1}$ are isomorphisms. Hence, the Five Lemma shows that $\Theta_{MM'}^i$ is an isomorphism. \square

The next result contains Theorem 5 from the introduction. A special case of it can be found in [2, Theorem 3].

Corollary 4.11. *Let M and M' be mini-max R -modules, and fix an index $i \geq 0$. If either M or M' is Matlis reflexive, then $\Theta_{MM'}^i$ is an isomorphism, so one has $\text{Ext}_R^i(M, M')^v = \text{Ext}_R^i(M, M')^\vee \cong \text{Tor}_i^R(M, M'^\vee)$, where $(-)^v = \text{Hom}_{\widehat{R}}(-, E)$.*

Proof. Apply Theorem 4.10 and Fact 1.18. \square

The next example shows that the modules $\text{Ext}_R^i(L, L')^\vee$ and $\text{Tor}_i^R(L, L'^\vee)$ are not isomorphic in general.

Example 4.12. Assume that R is not complete. We have $\text{Ann}_R(E) = 0$, so the ring $R/\text{Ann}_R(E) \cong R$ is not complete, by assumption. Thus, Fact 1.18 implies that E is not Matlis reflexive, that is, the biduality map $\delta_E: E \hookrightarrow E^{\vee\vee}$ is not an isomorphism. Since $E^{\vee\vee}$ is injective, we have $E^{\vee\vee} \cong E \oplus J$ for some non-zero

injective R -module J . The uniqueness of direct sum decompositions of injective R -modules implies that $E^{\vee\vee} \not\cong E$. This provides the second step below:

$$\mathrm{Hom}_R(E, E)^\vee \cong E^{\vee\vee} \not\cong E \cong E \otimes_R \widehat{R} \cong E \otimes_R E^\vee.$$

The third step is from Lemma 1.4(a), and the remaining steps are standard.

5. VANISHING OF EXT AND TOR

In this section we describe the sets of associated primes of $\mathrm{Hom}_R(A, M)$ and attached primes of $A \otimes_R M$ over \widehat{R} . The section concludes with some results on the related topic of vanishing for $\mathrm{Ext}_R^i(A, M)$ and $\mathrm{Tor}_i^R(A, M)$.

Associated and Attached Primes.

The following is dual to the notion of associated primes of noetherian modules; see, e.g., [11] or [12, Appendix to §6] or [14].

Definition 5.1. Let A be an artinian R -module. A prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ is *attached* to A if there is a submodule $A' \subseteq A$ such that $\mathfrak{p} = \mathrm{Ann}_R(A/A')$. We let $\mathrm{Att}_R(A)$ denote the set of prime ideals attached to A .

Lemma 5.2. Let A be an artinian R -module such that $R/\mathrm{Ann}_R(A)$ is complete, and let N be a noetherian R -module. There are equalities

$$\begin{aligned} \mathrm{Supp}_R(A^\vee) &= \cup_{\mathfrak{p} \in \mathrm{Ass}_R(A^\vee)} V(\mathfrak{p}) = \cup_{\mathfrak{p} \in \mathrm{Att}_R(A)} V(\mathfrak{p}) \\ \mathrm{Att}_R(N^\vee) &= \mathrm{Ass}_R(N) \\ \mathrm{Att}_R(A) &= \mathrm{Ass}_R(A^\vee). \end{aligned}$$

Proof. The R -module A^\vee is noetherian by Lemma 1.15(c), so the first equality is standard, and the second equality follows from the fourth one. The third equality is from [16, (2.3) Theorem]. This also explains the second step in the next sequence

$$\mathrm{Att}_R(A) = \mathrm{Att}_R(A^{\vee\vee}) = \mathrm{Ass}_R(A^\vee)$$

since A^\vee is noetherian. The first step in this sequence follows from the fact that A is Matlis reflexive; see Fact 1.18. \square

The next proposition can also be deduced from a result of Melkersson and Schenzel [13, Proposition 5.2].

Proposition 5.3. Let A and L be R -modules such that $\mu_R^0(L) < \infty$ and A is artinian. Then

$$\mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, L)) = \mathrm{Ass}_{\widehat{R}}(A^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) = \mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee).$$

Proof. The assumption $\mu_R^0(L) < \infty$ implies that $\Gamma_{\mathfrak{m}}(L)$ is artinian. This implies that $\Gamma_{\mathfrak{m}}(L)^\vee$ is a noetherian \widehat{R} -module, so a result of Bourbaki [4, IV 1.4 Proposition 10] provides the third equality in the next sequence; see also [5, Exercise 1.2.27]:

$$\begin{aligned} \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, L)) &= \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(L))) \\ &= \mathrm{Ass}_{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee, A^\vee)) \\ &= \mathrm{Ass}_{\widehat{R}}(A^\vee) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) \\ &= \mathrm{Att}_{\widehat{R}}(A) \cap \mathrm{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee). \end{aligned}$$

The remaining equalities are from Lemmas 1.5(b), 2.1(a), and 5.2, respectively. \square

Corollary 5.4. *Let M and M' be mini-max R -modules such that the quotient $R/(\text{Ann}_R(M) + \text{Ann}_R(M'))$ is complete.*

- (a) *For each index $i \geq 0$, one has $\text{Ext}_R^i(M, M') \cong \text{Ext}_R^i(M'^\vee, M^\vee)$.*
- (b) *If M' is noetherian, then*

$$\text{Ass}_{\widehat{R}}(\text{Hom}_R(M, M')) = \text{Att}_{\widehat{R}}(M'^\vee) \cap \text{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(M^\vee)^\vee).$$

Proof. (a) The first step in the next sequence comes from Theorem 2.8(a):

$$\text{Ext}_R^i(M, M') \cong \text{Ext}_R^i(M, M')^{\vee\vee} \cong (\text{Tor}_i^R(M, M'^\vee))^\vee \cong \text{Ext}_R^i(M'^\vee, M^\vee).$$

The remaining steps are from Theorem 4.10 and Remark 1.9, respectively.

- (b) This follows from the case $i = 0$ in part (a) because of Proposition 5.3. \square

Proposition 5.5. *Let A and L be R -modules such that A is artinian and $\beta_0^R(L) < \infty$. Then*

$$\text{Att}_{\widehat{R}}(A \otimes_R L) = \text{Ass}_{\widehat{R}}(A^\vee) \cap \text{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee) = \text{Att}_{\widehat{R}}(A) \cap \text{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee).$$

Proof. Theorem 3.1 implies that $A \otimes_R L$ is artinian. Hence, we have

$$\text{Hom}_{\widehat{R}}(A \otimes_R L, E) \cong \text{Hom}_R(A \otimes_R L, E) \cong \text{Hom}_R(A, L^\vee)$$

by Lemma 1.5(a), and this explains the second step in the next sequence:

$$\text{Att}_{\widehat{R}}(A \otimes_R L) = \text{Ass}_{\widehat{R}}(\text{Hom}_{\widehat{R}}(A \otimes_R L, E)) = \text{Ass}_{\widehat{R}}(\text{Hom}_R(A, L^\vee))$$

The first step is from Lemma 5.2. Since $\mu_R^0(L^\vee) < \infty$ by Lemma 1.13(b), we obtain the desired equalities from Proposition 5.3. \square

Next, we give an alternate description of the module $\Gamma_{\mathfrak{m}}(L)^\vee$ from the previous results. See Lemma 5.2 for a description of its support.

Remark 5.6. Let L be an R -module. There is an isomorphism $\Gamma_{\mathfrak{m}}(L)^\vee \cong \widehat{L}^\vee$. In particular, given a noetherian R -module N , one has $\Gamma_{\mathfrak{m}}(N^\vee)^\vee \cong \widehat{R} \otimes_R N$. When R is Cohen-Macaulay with a dualizing module D , Grothendieck's local duality theorem implies that $\Gamma_{\mathfrak{m}}(N)^\vee \cong \widehat{R} \otimes_R \text{Ext}_R^{\dim(R)}(N, D)$; see, e.g., [5, Theorem 3.5.8]. A similar description is available when R is not Cohen-Macaulay, provided that it has a dualizing complex; see [8, Chapter V, §6].

Vanishing of Hom and Tensor Product.

For the next result note that if L is noetherian, then the conditions on $\mu_R^0(L)$ and $R/(\text{Ann}_R(A) + \text{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$ are automatically satisfied. Also, the example $\text{Hom}_R(E, E) \cong R$ when R is complete shows the necessity of the condition on $R/(\text{Ann}_R(A) + \text{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$.

Proposition 5.7. *Let A be an artinian R -module. Let L be an R -module such that $R/(\text{Ann}_R(A) + \text{Ann}_R(\Gamma_{\mathfrak{m}}(L)))$ is artinian and $\mu_R^0(L) < \infty$. Then $\text{Hom}_R(A, L) = 0$ if and only if $A = \mathfrak{m}A$ or $\Gamma_{\mathfrak{m}}(L) = 0$.*

Proof. If $\Gamma_{\mathfrak{m}}(L) = 0$, then we are done by Lemma 1.5(b), so assume that $\Gamma_{\mathfrak{m}}(L) \neq 0$. Theorem 2.2 and Lemma 1.21 show that $\text{Hom}_R(A, L)$ has finite length. Thus Proposition 5.3 implies that $\text{Hom}_R(A, L) \neq 0$ if and only if $\mathfrak{m}\widehat{R} \in \text{Ass}_{\widehat{R}}(A^\vee)$, that is, if and only if $\text{depth}_{\widehat{R}}(A^\vee) = 0$. Lemma 1.13(c) shows that $\text{depth}_{\widehat{R}}(A^\vee) = 0$ if and only if $\mathfrak{m}\widehat{R}A \neq A$, that is, if and only if $\mathfrak{m}A \neq A$. \square

For the next result note that the conditions on L are satisfied when L is artinian.

Proposition 5.8. *Let A be an artinian R -module, and let L be an \mathfrak{m} -torsion R -module. The following conditions are equivalent:*

- (i) $A \otimes_R L = 0$;
- (ii) either $A = \mathfrak{m}A$ or $L = \mathfrak{m}L$; and
- (iii) either $\text{depth}_R(A^\vee) > 0$ or $\text{depth}_R(L^\vee) > 0$.

Proof. (i) \iff (ii) If $A \otimes_R L = 0$, then we have

$$0 = \text{len}_R(A \otimes_R L) \geq \beta_0^R(A)\beta_0^R(L)$$

so either $\beta_0^R(A) = 0$ or $\beta_0^R(L) = 0$, that is $A/\mathfrak{m}A = 0$ or $L/\mathfrak{m}L = 0$. Conversely, if $A/\mathfrak{m}A = 0$ or $L/\mathfrak{m}L = 0$, then we have either $\beta_0^R(A) = 0$ or $\beta_0^R(L) = 0$, so Theorem 3.8 implies that $\text{len}_R(A \otimes_R L) = 0$.

The implication (ii) \iff (iii) is from Lemma 1.13(c). \square

The next result becomes simpler when L is artinian, as $\Gamma_{\mathfrak{m}}(L) = L$ in this case.

Theorem 5.9. *Let A and L be R -modules such that A is artinian and $\mu_R^0(L) < \infty$. The following conditions are equivalent:*

- (i) $\text{Hom}_R(A, L) = 0$;
- (ii) $\text{Hom}_R(A, \Gamma_{\mathfrak{m}}(L)) = 0$;
- (iii) $\text{Hom}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee, A^\vee) = 0$;
- (iv) there is an element $x \in \text{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))$ such that $A = xA$;
- (v) $\text{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))A = A$;
- (vi) $\text{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L))$ contains a non-zero-divisor for A^\vee ; and
- (vii) $\text{Att}_{\widehat{R}}(A) \cap \text{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L)^\vee) = \emptyset$.

Proof. The equivalence (i) \iff (ii) is from Lemma 1.5(b). The equivalence (ii) \iff (vii) follows from Proposition 5.3, and the equivalence (ii) \iff (iii) follows from Lemma 2.1(a). The equivalence (iv) \iff (vi) follows from the fact that the map $A \xrightarrow{x} A$ is surjective if and only if the map $A^\vee \xrightarrow{x} A^\vee$ is injective. The equivalence (v) \iff (vi) follows from Lemma 1.13, parts (c) and (e).

The module $\Gamma_{\mathfrak{m}}(L)$ is artinian as $\mu_R^0(L) < \infty$. Since A^\vee and $\Gamma_{\mathfrak{m}}(L)^\vee$ are noetherian over \widehat{R} , the equivalence (iii) \iff (vi) is standard; see [5, Proposition 1.2.3]. \square

As with Theorem 5.9, the next result simplifies when L is noetherian. Also, see Remark 5.6 for some perspective on the module $\Gamma_{\mathfrak{m}}(L^\vee)^\vee$.

Corollary 5.10. *Let A be a non-zero artinian R -module, and let L be an R -module such that $\beta_0^R(L) < \infty$. The following conditions are equivalent:*

- (i) $A \otimes_R L = 0$;
- (ii) $\text{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee))A = A$;
- (iii) there is an element $x \in \text{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee))$ such that $xA = A$;
- (iv) $\text{Ann}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee))$ contains a non-zero-divisor for A^\vee ; and
- (v) $\text{Att}_{\widehat{R}}(A) \cap \text{Supp}_{\widehat{R}}(\Gamma_{\mathfrak{m}}(L^\vee)^\vee) = \emptyset$.

Proof. For an artinian R -module A' , one has $\text{Att}_{\widehat{R}}(A') = \emptyset$ if and only if $A' = 0$ by Lemma 5.2. Thus, Proposition 5.5 explains the equivalence (i) \iff (v); see [14, Corollary 2.3]. Since one has $A \otimes_R L = 0$ if and only if $(A \otimes_R L)^\vee = 0$, the isomorphism $(A \otimes_R L)^\vee \cong \text{Hom}_R(A, L^\vee)$ from Remark 1.9 in conjunction with Theorem 5.9 shows that the conditions (i)–(iv) are equivalent. \square

Depth and Vanishing.

Proposition 5.11. *Let A and L be R -modules such that A is artinian. Then $\text{Ext}_R^i(A, L) = 0$ for all $i < \text{depth}_R(L)$.*

Proof. Let J be a minimal R -injective resolution of L , and let $i < \text{depth}_R(L)$. It follows that $\text{Ext}_R^i(k, L) = 0$, that is $\mu_R^i(L) = 0$, so the module E does not appear as a summand of J^i . As in the proof of Theorem 2.2, this implies that $\text{Hom}_R(A, J)^i = 0$, so $\text{Ext}_R^i(A, L) = 0$. \square

The next example shows that, in Proposition 5.11 one may have $\text{Ext}_R^i(A, L) = 0$ when $i = \text{depth}_R(L)$. See also equation (5.14.1).

Example 5.12. Assume that $\text{depth}(R) \geq 1$. Then $\mathfrak{m}E = E$ by Lemma 1.13(c), so Lemma 2.10 implies that

$$\text{Ext}_R^0(E, k) \cong \text{Hom}_R(E, k) \cong \text{Hom}_R(E/\mathfrak{m}E, k) = 0$$

even though $\text{depth}_R(k) = 0$.

Proposition 5.13. *Let A and L be R -modules such that A is artinian. Then for all $i < \text{depth}_R(L^\vee)$ one has $\text{Tor}_i^R(A, L) = 0$.*

Proof. When $i < \text{depth}_R(L^\vee)$, one has $\text{Tor}_i^R(A, L)^\vee \cong \text{Ext}_R^i(A, L^\vee) = 0$ by Remark 1.9 and Proposition 5.11, so $\text{Tor}_i^R(A, L) = 0$. \square

Theorem 5.14. *Let A and A' be artinian R -modules, and let N and N' be noetherian R -modules. Then one has*

$$(5.14.1) \quad \text{depth}_{\widehat{R}}(\text{Ann}_{\widehat{R}}(A'); A^\vee) = \inf\{i \geq 0 \mid \text{Ext}_R^i(A, A') \neq 0\}$$

$$(5.14.2) \quad \text{depth}_R(\text{Ann}_R(N'); A^\vee) = \inf\{i \geq 0 \mid \text{Ext}_R^i(A, N'^\vee) \neq 0\}$$

$$(5.14.3) \quad \text{depth}_R(\text{Ann}_R(N'); N) = \inf\{i \geq 0 \mid \text{Ext}_R^i(N^\vee, N'^\vee) \neq 0\}.$$

Proof. We verify equation (5.14.1) first. For each index i , Theorem 4.3 implies that

$$\text{Ext}_R^i(A, A') \cong \text{Ext}_{\widehat{R}}^i(A'^\vee, A^\vee).$$

Since A^\vee and A'^\vee are noetherian over \widehat{R} , this explains the first equality below:

$$\inf\{i \geq 0 \mid \text{Ext}_R^i(A, A') \neq 0\} = \text{depth}_{\widehat{R}}(\text{Ann}_{\widehat{R}}(A'^\vee); A^\vee) = \text{depth}_{\widehat{R}}(\text{Ann}_{\widehat{R}}(A'); A^\vee).$$

The second equality is standard since $A'^\vee = \text{Hom}_{\widehat{R}}(A', E)$ by Lemma 1.5(a).

Next, we verify equation (5.14.2). Since N'^\vee is artinian, equation (5.14.1) shows that we need only verify that

$$(5.14.4) \quad \text{depth}_{\widehat{R}}(\text{Ann}_{\widehat{R}}(N'^\vee); A^\vee) = \text{depth}_R(\text{Ann}_R(N'); A^\vee).$$

For this, we compute as follows:

$$\widehat{R} \otimes_R N' \xrightarrow{(1)} \text{Hom}_{\widehat{R}}(\text{Hom}_{\widehat{R}}(\widehat{R} \otimes_R N', E), E) \xrightarrow{(2)} \text{Hom}_{\widehat{R}}(N'^\vee, E).$$

Step (1) follows from the fact that $\widehat{R} \otimes_R N'$ is noetherian (hence, Matlis reflexive) over \widehat{R} , and step (2) is from Hom-tensor adjointness. This explains step (4) below:

$$\text{Ann}_{\widehat{R}}(N'^\vee) \xrightarrow{(3)} \text{Ann}_{\widehat{R}}(\text{Hom}_{\widehat{R}}(N'^\vee, E)) \xrightarrow{(4)} \text{Ann}_{\widehat{R}}(\widehat{R} \otimes_R N') \xrightarrow{(5)} \text{Ann}_R(N')\widehat{R}.$$

Steps (3) and (5) are standard. This explains step (6) in the next sequence:

$$\text{depth}_{\widehat{R}}(\text{Ann}_{\widehat{R}}(N'^\vee); A^\vee) \xrightarrow{(6)} \text{depth}_{\widehat{R}}(\text{Ann}_R(N')\widehat{R}; A^\vee) \xrightarrow{(7)} \text{depth}_R(\text{Ann}_R(N'); A^\vee).$$

Step (7) is explained by the following, where step (8) is standard, and step (9) is a consequence of Hom-tensor adjointness:

$$\begin{aligned} \text{Ext}_{\widehat{R}}^i(\widehat{R}/\text{Ann}_R(N')\widehat{R}, A^\vee) &\stackrel{(8)}{\cong} \text{Ext}_{\widehat{R}}^i(\widehat{R} \otimes_R (R/\text{Ann}_R(N')), A^\vee) \\ &\stackrel{(9)}{\cong} \text{Ext}_R^i(R/\text{Ann}_R(N'), A^\vee). \end{aligned}$$

This establishes equation (5.14.4) and thus equation (5.14.2).

Equation (5.14.3) follows from (5.14.2) because we have

$$\text{depth}_R(\text{Ann}_R(N'); N^{\vee\vee}) = \text{width}_R(\text{Ann}_R(N'); N^\vee) = \text{depth}_R(\text{Ann}_R(N'); N)$$

by Lemma 1.13(a). \square

Corollary 5.15. *Let A and A' be artinian R -modules, and let N and N' be noetherian R -modules. Then*

$$(5.15.1) \quad \text{depth}_{\widehat{R}}(\text{Ann}_{\widehat{R}}(A'); A^\vee) = \inf\{i \geq 0 \mid \text{Tor}_i^R(A, A'^\vee) \neq 0\}$$

$$(5.15.2) \quad \text{depth}_R(\text{Ann}_R(N'); A^\vee) = \inf\{i \geq 0 \mid \text{Tor}_i^R(A, N') \neq 0\}$$

$$(5.15.3) \quad \text{depth}_R(\text{Ann}_R(N'); N) = \inf\{i \geq 0 \mid \text{Tor}_i^R(N^\vee, N') \neq 0\}.$$

Proof. We verify equation (5.15.1); the others are verified similarly.

Since $\text{Ext}_R^i(A, A') \neq 0$ if and only if $\text{Hom}_{\widehat{R}}(\text{Ext}_R^i(A, A'), E) \neq 0$, the isomorphism $\text{Hom}_{\widehat{R}}(\text{Ext}_R^i(A, A'), E) \cong \text{Tor}_i^R(A, A'^\vee)$ from Corollary 4.9 shows that

$$\inf\{i \geq 0 \mid \text{Ext}_R^i(A, A') \neq 0\} = \inf\{i \geq 0 \mid \text{Tor}_i^R(A, A'^\vee) \neq 0\}.$$

Thus equation (5.15.1) follows from (5.14.1). \square

6. EXAMPLES

This section contains some explicit computations of Ext and Tor for the classes of modules discussed in this paper. Our first example shows that $\text{Ext}_R^i(A, A')$ need not be mini-max over R .

Example 6.1. Let k be a field, and set $R = k[X_1, \dots, X_d]_{(X_1, \dots, X_d)}$. We show that $\text{Hom}_R(E, E) \cong \widehat{R}$ is not mini-max over R . Note that R is countably generated over k , and $\widehat{R} \cong k[[X_1, \dots, X_d]]$ is not countably generated over k . So, \widehat{R} is not countably generated over R . Also, every artinian R -module A is a countable union of the finite length submodules $(0 :_A \mathfrak{m}^n)$, so A is countably generated. It follows that every mini-max R -module is also countably generated. Since \widehat{R} is not countably generated, it is not mini-max over R .

Our next example describes $\text{Ext}_R^i(A, A')$ for some special cases.

Example 6.2. Assume that $\text{depth}(R) \geq 1$, and let A be an artinian R -module. Let $x \in \mathfrak{m}$ be an R -regular element. The map $E \xrightarrow{x} E$ is surjective since E is divisible, and the kernel $(0 :_E x)$ is artinian, being a submodule of E . Using the injective resolution $0 \rightarrow E \xrightarrow{x} E \rightarrow 0$ for $(0 :_E x)$, one can check that

$$\text{Ext}_R^i(A, (0 :_E x)) \cong \begin{cases} (0 :_{A^\vee} x) & \text{if } i = 0 \\ A^\vee/xA^\vee & \text{if } i = 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

For instance, in the case $A = (0 :_E x)$, the isomorphism $(0 :_E x)^\vee \cong \widehat{R}/x\widehat{R}$ implies

$$\mathrm{Ext}_R^i((0 :_E x), (0 :_E x)) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = 0, 1 \\ 0 & \text{if } i \neq 0, 1. \end{cases}$$

On the other hand, if x, y is an R -regular sequence, then $(0 :_E y)^\vee \cong \widehat{R}/y\widehat{R}$; it follows that x is $(0 :_E y)^\vee$ -regular, so one has

$$\mathrm{Ext}_R^i((0 :_E y), (0 :_E x)) \cong \begin{cases} \widehat{R}/(x, y)\widehat{R} & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

The next example shows that $\mathrm{Ext}_R^i(A, N)$ need not be mini-max over R .

Example 6.3. Assume that R is Cohen-Macaulay with $d = \dim(R)$, and let A be an artinian R -module. Assume that R admits a dualizing (i.e., canonical) module D . (For instance, this is so when R is Gorenstein, in which case $D = R$.) A minimal injective resolution of D has the form

$$J = 0 \rightarrow \coprod_{\mathrm{ht}(\mathfrak{p})=0} E_R(R/\mathfrak{p}) \rightarrow \cdots \rightarrow \coprod_{\mathrm{ht}(\mathfrak{p})=d-1} E_R(R/\mathfrak{p}) \rightarrow E \rightarrow 0.$$

In particular, we have $\Gamma_{\mathfrak{m}}(J) = (0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow E \rightarrow 0)$ where the copy of E occurs in degree d . Since $\mathrm{Hom}_R(A, J) \cong \mathrm{Hom}_R(A, \Gamma_{\mathfrak{m}}(J))$, it follows that

$$\mathrm{Ext}_R^i(A, D) \cong \begin{cases} A^\vee & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Assume that $d \geq 1$, and let $x \in \mathfrak{m}$ be an R -regular element. It follows that the map $D \xrightarrow{x} D$ is injective, and the cokernel D/xD is noetherian. Consider the exact sequence $0 \rightarrow D \xrightarrow{x} D \rightarrow D/xD \rightarrow 0$. The long exact sequence associated to $\mathrm{Ext}_R^i(A, -)$ shows that

$$\mathrm{Ext}_R^i(A, D/xD) \cong \begin{cases} (0 :_{A^\vee} x) & \text{if } i = d-1 \\ A^\vee/xA^\vee & \text{if } i = d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

As in Example 6.2, we have $(0 :_E x)^\vee \cong \widehat{R}/x\widehat{R}$ and

$$\mathrm{Ext}_R^i((0 :_E x), D/xD) \cong \begin{cases} \widehat{R}/x\widehat{R} & \text{if } i = d-1, d \\ 0 & \text{if } i \neq d-1, d. \end{cases}$$

Also, if x, y is an R -regular sequence, then $(0 :_E y)^\vee \cong \widehat{R}/y\widehat{R}$ and

$$\mathrm{Ext}_R^i((0 :_E y), D/xD) \cong \begin{cases} \widehat{R}/(x, y)\widehat{R} & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Next, we show that $\mathrm{Tor}_i^R(A, A')$ need not be noetherian over R or \widehat{R} .

Example 6.4. Assume that R is Gorenstein and complete with $d = \dim(R)$. (Hence $D = R$ is a dualizing R -module.) Given two artinian R -modules A and A' , Theorem 3.1 implies that $\mathrm{Tor}_i^R(A, A')$ is artinian, hence Matlis reflexive for

each index i , since R is complete. This explains the first isomorphism below, and Remark 1.9 provides the second isomorphism:

$$\mathrm{Tor}_i^R(A, E) \cong \mathrm{Tor}_i^R(A, E)^{\vee\vee} \cong \mathrm{Ext}_R^i(A, E^\vee)^\vee \cong \mathrm{Ext}_R^i(A, R)^\vee \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Example 6.3 explains the fourth isomorphism. Assume that $d \geq 1$, and let $x \in \mathfrak{m}$ be an R -regular element. Then $(0 :_E x)^\vee \cong R/xR$, so Example 6.3 implies that

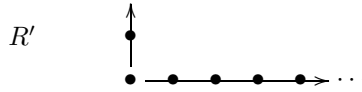
$$\begin{aligned} \mathrm{Tor}_i^R(A, (0 :_E x)) &\cong \mathrm{Ext}_R^i(A, (0 :_E x)^\vee)^\vee \cong \begin{cases} A/xA & \text{if } i = d-1 \\ (0 :_A x) & \text{if } i = d \\ 0 & \text{if } i \neq d-1, d \end{cases} \\ \mathrm{Tor}_i^R((0 :_E x), (0 :_E x)) &\cong \begin{cases} (0 :_E x) & \text{if } i = d-1, d \\ 0 & \text{if } i \neq d-1, d. \end{cases} \end{aligned}$$

On the other hand, if x, y is an R -regular sequence, then

$$\mathrm{Tor}_i^R((0 :_E y), (0 :_E x)) \cong \begin{cases} (R/(x, y)R)^\vee \cong E_{R/(x, y)R}(k) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Lastly, we provide an explicit computation of $E \otimes_R E$.

Example 6.5. Let k be a field and set $R = k[[X, Y]]/(XY, Y^2)$. This is the completion of the multi-graded ring $R' = k[X, Y]/(XY, Y^2)$ with homogeneous maximal ideal $\mathfrak{m}' = (X, Y)R'$. The multi-graded structure on R' is represented in the following diagram:



where each bullet represents the corresponding monomial in R' . It follows that $E \cong E_{R'}(k) \cong k[X^{-1}] \oplus kY^{-1}$ with graded module structure given by the formulas

$$\begin{aligned} X \cdot 1 &= 0 & X \cdot X^{-n} &= X^{1-n} & X \cdot Y^{-1} &= 0 \\ Y \cdot 1 &= 0 & Y \cdot Y^{-1} &= 1 & Y \cdot X^{-n} &= 0 \end{aligned}$$

for $n \geq 1$. Using this grading, one can show that $\mathfrak{m}E = \mathfrak{m}'E \cong k[X^{-1}]$ and $\mathfrak{m}^2E = \mathfrak{m}E$. These modules are represented in the next diagrams:



It follows that $E/\mathfrak{m}E \cong k$, so Lemma 3.7 implies that

$$E \otimes_R E \cong (E/\mathfrak{m}E) \otimes_R (E/\mathfrak{m}E) \cong k \otimes_R k \cong k.$$

A similar computation shows the following: Fix positive integers a, b, c such that $c > b$, and consider the ring $S = k[[X, Y]]/(X^a Y^b, Y^c)$ with maximal ideal \mathfrak{n} and $E_S = E_S(k)$. Then $\mathfrak{n}^{c-b}E_S = \mathfrak{n}^{c-b+1}E_S$ and we get the following:

$$\begin{aligned} E_S/\mathfrak{n}^{c-b}E_S &\cong S/(X^a, Y^{c-b})S \cong k[X, Y]/(X^a, Y^{c-b}) \\ E_S \otimes_S E_S &\cong (E_S/\mathfrak{n}^{c-b}E_S) \otimes_S (E_S/\mathfrak{n}^{c-b}E_S) \cong S/(X^a, Y^{c-b})S. \end{aligned}$$

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